BEHAVIOUR OF THE RISE AND FALL CHARACTERISTICS OF THE CHANNEL DISCHARGE FUNCTION FOR THE GENERALIZED LIGHTNING TRAVELING CURRENT SOURCE RETURN STROKE MODEL

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Abstract: The channel discharge function in the generalized lightning travelling current source return stroke model (GTCS) in a complex and time domain is analyzed. Heidler's form of the channel-base current function is accepted. The expression of the initial channel charge distribution function is proposed. First, using this form of the charge distribution function it is shown that the previously established return stroke models, the Bruce-Golde (BG), the Traveling Current Source - (TCS), the Diendorfer-Uman (DU) and the Thottappillil-Uman (TU) models can be derived as the special cases of the GTCS. Second, if the general case is treated , i.e., the values of the parameters in the initial channel charge distribution function are arbitrary (independent of the channel-base current parameters), certain limits of their values can be deduced regarding the general properties of the channel discharge function in the GTCS. Third, for this case the simple analytical forms of the channel discharge function in the time domain for a few microseconds of the discharge (discharge risetime) at the observed channel altitude are derived. Since the risetime of the channel discharge function defines the magnitude and the risetime of the radiated lightning electromagnetic pulse (LEMP) it turns out that the measuring of the LEMP at various distances can provide the values of the function parameters. This will be the "key" for the further examination of the gaseous-dynamic processes in the channel during the return stroke phase.

1. Introduction

Based on the shortcomings of the existing traveling current source return stroke models (the BG [1], the TCS [2], the DU [3] and the TU [4] model), the new GTCS return stroke model has been developed [5, 6]. It eliminates completely all disadvantages of the mentioned models concerning the current discontinuities and the discontinuities of the current derivative at the place of the return stroke wave-front [6]. As a result of the suitable adopted channel charge distribution function the dynamics of the internal channel processes can be partially examined during the return stroke. Moreover, using a two-layer cylindrical model of the channel it is possible to derive the simple connection between the channel time-discharge constant and the channel discharge function. [6]. On the other hand, it represents the generalization of the traveling current source models taking into account that the BG, the TCS, the DU and the TU models can be easily carried out from the new model as its special cases.

In the GTCS model the assumption of the existence of the traveling current source is adapted from the TCS model. Although the current reflection from the bottom of the channel can be taken into account we shall neglect it due to the simplicity of the mathematical derivations. The channel-base current at the striking point $i_0(t)$ and the initial charge distribution along the channel $q_0^{\prime}(z)$ are considered as known. Hence, the channel charge at some altitude z, at some instant of time t, is

$$q^{\prime}(z,t) = q_0^{\prime}(z) f(z, t - z/v) \ , \quad t \geq z/v \ , \tag{1}$$

where f represents the channel discharge function. The channel discharge function is given by

$$f(z, t - z/v) \ = \ 1 + \int_0^{t-z/v} f_1(z, \xi)\, d\xi \ . \tag{2}$$

Function f_1 can be obtained using Fourier's inversion formula (denoted as $\mathscr{F}^{-1}$)

$$f_1(z,u) = \partial f(z,u)/\partial u = \mathscr{F}^{-1}\left[\mathrm{F}_1(z,s) \right] \ , \qquad \mathrm{F}_1(z,s) = \mathrm{I}_0(s) / Q_0^{\prime}(z, s/v^*) \ , \tag{3}$$

where $s = j\omega$, $u = t - z/v$, $(u \geq 0)$, $I_0 = \mathscr{F}(i_0)$ and $Q_0^{\prime} = \mathscr{F}(q_0^{\prime})$ are the Fourier transforms of the channel-base current and the initial channel charge distribution, respectively. Since there is no loss of charge through other processes (for example air discharges) these two functions are connected by the charge conservation law

$$\int_0^{\infty} i_0(t)\, dt = \int_0^{\infty} q_0^{\prime}(z)\, dz \ , \tag{4}$$

where, for the sake of the simplicity of the theoretical considerations, it is supposed that the length of the channel is infinite. In accordance with this assumption the duration of the return stroke is assessed to be also infinite.

The current at some altitude is given by [5, 6]:

$$i(z,t) = \int_z^{h_{az}(z,t)} q_0^{\prime}(\xi) \frac{\partial}{\partial t} f(z, t - \xi/v^* + z/c)\, d\xi \ , \tag{5}$$

where $v^* = vc/(v+c)$ is the so-called reduced return stroke velocity and $h_{az} = v^*(t + z/c)$. In the above expressions it is assumed that the upward return stroke velocity (v) as well as the downward discharge current speed (c) are constant. If they are the functions of altitude their average values should be used as it is done in the modified Diendorfer-Uman model (MDU) [7].

According to the assumed mechanism of the channel discharge it is possible to deduce four properties of the channel discharge function. Taking into account that the charge in the channel at altitude z starts to discharge at the instant of time $t = z/v$ it follows from Eq.(1)

$$f(z, u=0) = 1 \ , \quad u = t - z/v \, . \tag{6}$$

Similarly, after the discharging process there should be no net charge along the channel. Therefore one obtains

$$f(z, u \to \infty) = 0 \ . \tag{7}$$

The third and the fourth features of the channel discharge function follow from the assumption that the laterally deposited charge along the corona sheath below the return stroke wave-front diminishes monotony to the zero

$$f(z, u \geq 0) \geq 0 \ , \tag{8}$$

$$\partial f(z,u)/\partial u \,\big|_{u \geq 0} \ \leq \ 0 \ . \tag{9}$$

Basically, the measurements of the remote LEMP are based on the measurements of magnetic and electric field derivative (the LEMP is obtained by the integration of the raw data). Since the measured values of the field derivative contain no discontinuities it turns out that the derivative of the channel discharge function at the time onset ($u = 0$) must be zero, Eq.(9).

2. The channel-base current and the initial channel charge distribution function

We have accepted the following form of the current function at the striking point [2, 9]

$$i_0(t) = (\mathrm{I}_0/\eta)\, l(t)$$
$$l(t) = [k_s^{\,n}/(1+k_s^{\,n})]\exp(-t/\tau_2)\ ,\ k_s = t/\tau_1\ . \tag{10}$$

The values of the parameters in Eq.(10) can be determined using the measurements [8] and the graphical method [9]. As it is discussed in [9] and [10] this form of the current function is very convenient for lightning calculations.

For the initial charge distribution function the following form is assumed

$$q_0^{/}(z) = -Q_{01}^{/}\left[g(z) + \lambda_{\mathrm{d1}}(z)\frac{dg}{dz} + \lambda_{\mathrm{d2}}(z)\frac{d^2g}{dz^2} + \dots\right],$$
$$g(z) = [z^{\,m}/(z^{\,m}+\lambda_1^m)]\exp(-z/\lambda_2)\ , \tag{11}$$

where $Q_{01}^{/}$, λ_1, λ_2, m are the channel charge distribution parameters. The minus sign in Eq.(11) corresponds to the negative charged lightning channel (hence $Q_{01}^{/}>0$). The examinations of the parameters in Eq.(11) have shown that λ_1, λ_2 and m determine the time-dependence of the channel discharge function. The parameters $\lambda_{\mathrm{d1}}(z), \lambda_{\mathrm{d2}}(z), \dots$ define the height-dependence of the channel discharge function. This form of the function is chosen because of two reasons: first, it enables (for special parameter values) the same results as previously established models (the BG, the TCS, the DU and the TU models). Second, the influence of the parameters on the time and the space behaviour of the channel discharge function is fully separated. Moreover the parameters $\lambda_{\mathrm{d1}}(z), \lambda_{\mathrm{d2}}(z), \dots$ do not change the total amount of charge along the channel, they change only the shape of the distribution function. Third, as it is shown in previous investigations [5, 6] this form of function is very flexible enabling the simulation of the uniform as well as the strongly nonuniform initial charge distribution.

The connection between the channel-base current and the charge distribution along lightning channel, i.e. between I_0 and $Q_{01}^{/}$ can be obtained from the charge conservation law, Eq.(4). Therefore we obtain

$$Q_{01}^{/} = (\mathrm{I}_0/\eta)\left(\int_0^\infty l(t)\,dt\right) / \left(\int_0^\infty g(z)\,dz\right)\ . \tag{12}$$

3. Analysis of the channel discharge function in the complex domain

The exact expression of the channel discharge function in the complex domain is necessary for the further analysis during rise- and falltime. Applying the Fourier transform to the expression of the channel-base current, Eq.(10) and on the expression of the initial channel charge distribution, Eq.(11) one obtains

$$I_0(s) = \mathscr{F}[i_0(t)] \ = \ (\mathrm{I}_0/\eta)\, L(s)\ ,$$
$$Q_0^{/}(s) = \mathscr{F}[q_0^{/}(z)] = -Q_{01}^{/}\, G(s)\,[1+\sum_{\mathrm{i}=1}^{\infty}\lambda_{\mathrm{di}}^{\mathrm{i}}\, s^{\,\mathrm{i}}]\ , \tag{13}$$

where $L(s) = \mathscr{F}[l(t)]$ and $G(s) = \mathscr{F}[g(z)]$. Using Eq.(3) follows the expression for F_1 in the complex domain

$$\mathrm{F}_1(s) = -\frac{\mathrm{I}_0}{\mathrm{Q}_{01}^{/}\eta}\ \frac{L(s)}{G(s/v^*)\,[1+\sum_{\mathrm{i}=1}^{\infty}\lambda_{\mathrm{di}}^{\mathrm{i}}\,(s/v^*)^{\mathrm{i}}]}\ . \tag{14}$$

Eq.(14) represents the general case of the channel discharge function in the GTCS in the complex domain. It means that the values of the parameters in Eqs (10) and (11) are arbitrary and in the general case they should be $n\neq m$, $\lambda_1\neq v^*\tau_1$, $\lambda_2\neq v^*\tau_2$, $\lambda_{\mathrm{di}}\neq 0$, $\mathrm{i}=1,2,\dots$ whereas the values of the parameters $Q_{01}^{/}$ and I_0 are connected by Eq.(12).

4. The behaviour of the channel discharge function in the time domain during risetime

We shall derive the approximative expressions of the channel discharge function in the time domain during the risetime (a few microseconds of the channel section discharge). The general case of the GTCS will be treated. These formulae can be used for further calculation of the lightning current along the channel using Eq.(5) as well as for further calculation of the radiated LEMP magnitude.

4.1 The behaviour of the channel discharge function for $\lambda_{\mathrm{di}} = 0$, i = 1,2,...

Suppose that all the channel discharge constants defining the height-dependence of the channel discharge function are equal to zero, i.e., $\lambda_{\mathrm{di}} = 0$, $\mathrm{i} = 1,2,\dots$ The analytical expressions of the channel discharge function in the time domain for the risetime ($u\to 0$) can be derived if the values of Fourier's transforms in Eq.(13) are calculated in the high frequency domain $s\to\infty$. From Eqs (10) and (11) it turns out that the terms $l(t)$ and $g(z)$ take the values

$$l(t)\ \cong\ t^{\,n}/\tau_1^n\ ,\quad t\to 0\ ;\qquad g(z)\ \cong\ z^{\,m}/\lambda_1^m\ ,\quad z\to 0\ . \tag{15}$$

Using the Fourier transform of the power function we have

$$\mathscr{F}(t^{\,n-1})\ =\ (n-1)!/s^{\,n}\ . \tag{16}$$

Applying it on Eq.(15) and replacing into Eqs (13) and (14) one obtains

$$\mathrm{F}_1(s\to\infty) \cong -\mathrm{A}\,\frac{n!}{m!}\,\frac{1}{s^{\,n-m}}\,\frac{\tau_1^{*m}}{\tau_1^n}\ . \tag{17}$$

where $\mathrm{A} = \mathrm{I}_0/(Q_{01}^{/}\eta\, v^*)\neq 1$ and $\tau_1^* = \lambda_1/v^*$. Returning into time domain by use of the inverse Fourier transform it follows

$$\mathrm{f}_1(u\to 0) \approx -\mathrm{A}\,\frac{n!}{m!}\,\frac{\tau_1^{*m}}{\tau_1^n}\,\frac{u^{\,n-m-1}}{(n-m-1)!}\ . \tag{18}$$

If one compares Eq.(18) with the feature of the channel discharge function $f_1(u=0) = 0$, Eq.(9) it follows that the values of the exponents in Eqs (10) and (11) should satisfy the inequality $n\geq m+1$. It is obvious that the current steepness (n) and charge distribution parameter (m) define the dynamics of the channel discharge during the risetime.

The substitution of Eq.(18) into Eq.(2) yields

$$\mathrm{f}(u\to 0) \cong 1 - \mathrm{A}\,\frac{n!}{m!}\,(\tau_1^{*m}/\tau_1^n)\,\frac{u^{\,n-m}}{(n-m)!}\ . \tag{19}$$

There is no height-dependence of the channel discharge function in this case. Therefore, this type of discharge function is convenient for the examination of the discharge processes in the small sections of the channel, maybe some tens of meters.

4.2 The behaviour of the channel discharge function for $\lambda_{\mathrm{d1}}\neq 0$ and $\lambda_{\mathrm{di}} = 0$, i = 2,3,...

Suppose that only the first channel discharge constant defining the height-dependence of the channel discharge function is not equal to zero i.e., $\lambda_{\mathrm{d1}}(z)\neq 0$. Applying similar a procedure as in the previous case one obtains

$$\mathrm{F}_1(z,s\to\infty) \cong -\mathrm{A}\,\frac{n!}{m!}\,\frac{1}{s^{\,n-m+1}}\,\frac{\tau_1^{*m}}{\tau_1^n\,\tau_{\mathrm{d1}}^*(z)}\ . \tag{20}$$

where $\tau_{\mathrm{d1}}^*(z) = \lambda_{\mathrm{d1}}(z)/v^*$. Using the inverse Fourier transform it follows

$$\mathrm{f}_1(z,u\to 0) \cong -\mathrm{A}\,\frac{n!}{m!}\,\frac{\tau_1^{*m}}{\tau_1^n\,\tau_{\mathrm{d1}}^*(z)}\,\frac{u^{\,n-m}}{(n-m)!}\ . \tag{21}$$

The condition $f_1(u=0) = 0$, given by Eq.(9) is satisfied if $n\geq m$. Using Eq.(2) one obtains

$$\mathrm{f}(z,u\to 0)\ \cong\ 1 - \mathrm{A}\,\frac{n!}{m!}\,\frac{\tau_1^{*m}}{\tau_1^n\,\tau_{\mathrm{d1}}^*(z)}\,\frac{u^{\,n-m+1}}{(n-m+1)!}\ . \tag{22}$$

It is reasonable to expect that the value of the parameter $\tau_{\mathrm{d1}}^*(z)$ increases with the channel height. This is shown in the DU model [3] where two values are used: the breakdown time discharge constant $\tau_{\mathrm{dbd}}^* = 0.6\,\mu s$ for the lower parts of the channel and the corona time discharge constant $\tau_{\mathrm{dco}}^* = 5\,\mu s$ for the upper parts of the channel. The similar result is obtained in the case of the TU model [4]. In this case the continuous function for the time discharge constant was obtained starting with the value $0.6\,\mu s$ at the channel-base ($z=0$) and increasing rapidly with the channel height.

4.3 The behaviour of the channel discharge function for $\lambda_{d1} \neq 0,\ \lambda_{d2} \neq 0,\ \lambda_{di} = 0,\ i = 3,4,...$

Let us examine the case when $\lambda_{d1} \neq 0$ and $\lambda_{d2} \neq 0$. Eq.(14) takes the form

$$F_1(z, s\to\infty) \cong -A \frac{n!}{m!} \frac{1}{s^{n-m+2}} \frac{\tau_1^{*m}}{\tau_1^n \tau_{d2}^{*2}(z)} , \qquad (23)$$

where $\tau_{d2}^*(z) = \lambda_{d2}(z)/v^*$ and the term containing λ_{d1} is neglected because $s\to\infty$. Returning into the time domain using the inverse Fourier transform it follows

$$f_1(z, u\to 0) \cong -A \frac{n!}{m!} \frac{\tau_1^{*m}}{\tau_1^n \tau_{d2}^{*2}(z)} \frac{u^{n-m+1}}{(n-m+1)!} . \qquad (24)$$

The condition $f_1(u{=}0) = 0$ is satisfied if $n \geq m-1$. The channel discharge function is given by

$$f(z, u\to 0) \cong 1 - A \frac{n!}{m!} \frac{\tau_1^{*m}}{\tau_1^n \tau_{d2}^{*2}(z)} \frac{u^{n-m+2}}{(n-m+2)!} . \qquad (25)$$

It is interesting to notice that the channel discharge function is not affected by parameter λ_{d1} although its value is not zero.

4.4 The behaviour of the channel discharge function for $\lambda_{dj} \neq 0,\ \lambda_{di} = 0,\ i > j$

In the similar manner one can derive the expression of the channel discharge function in the general case $\lambda_{dj} \neq 0$ if $\lambda_{di} = 0$, $i > j$. From Eqs (13) and (14) one obtains

$$F_1(z, s\to\infty) \cong -A \frac{n!}{m!} \frac{1}{s^{n-m+j}} \frac{\tau_1^{*m}}{\tau_1^n \tau_{dj}^{*j}(z)} , \qquad (26)$$

where $\tau_{dj}^*(z) = \lambda_{dj}(z)/v^*$ the influence of all terms containing λ_{di}, $i < j$ can be neglected because $s\to\infty$. Using the inverse Fourier transform it follows

$$f_1(z, u\to 0) \cong -A \frac{n!}{m!} \frac{\tau_1^{*m}}{\tau_1^n \tau_{dj}^{*j}(z)} \frac{u^{n-m+j-1}}{(n-m+j-1)!} . \qquad (27)$$

The condition $f_1(u{=}0) = 0$, is satisfied for $n \geq m-j+1$. Substituting Eq.(27) into Eq.(2) yields

$$f(z, u\to 0) \cong 1 - A \frac{n!}{m!} \frac{\tau_1^{*m}}{\tau_1^n \tau_{dj}^{*j}(z)} \frac{u^{n-m+j}}{(n-m+j)!} . \qquad (28)$$

5. The behaviour of the channel discharge function during falltime for the general case of the GTCS

The behaviour of the channel discharge function during falltime can be obtained from the calculation in the complex domain for $s\to 0$. Substituting this condition into Eq.(14) the approximative expression describing the behaviour of the "tail" of the channel discharge function can be derived

$$F_1(s) \cong -[I_0\, L(s)]/[Q_{01}^{/}\,\eta\, G(s/v^*)] , \quad s\to 0 . \qquad (29)$$

From Eq.(13) one obtains

$$\begin{aligned} G(s) &= \int_0^\infty \frac{z^m}{\lambda_1^m + z^m} \exp(-z/\lambda_2) \exp(-sz)\, dz \\ &\cong \int_0^\infty \frac{z^m}{\lambda_1^m + z^m} \exp(-z/\lambda_2)\, dz = \int_0^\infty g(z)\, dz , \ s\to 0 . \end{aligned} \qquad (30)$$

$$\begin{aligned} L(s) &= \int_0^\infty \frac{t^n}{\tau_1^n + t^n} \exp(-t/\tau_2) \exp(-st)\, dt \\ &\cong \int_0^\infty \frac{t^n}{\tau_1^n + t^n} \exp(-t/\tau_2)\, dt = \int_0^\infty l(t)\, dt , \quad s\to 0 . \end{aligned} \qquad (31)$$

Substituting Eqs (30) and (31) into Eq.(29), using Eq.(12) we have

$$F_1(s\to 0) \cong -\frac{I_0}{Q_{01}^{/}\eta} \frac{\int_0^\infty l(t)\, dt}{\int_0^\infty g(z)\, dz} = -1 , \qquad (32)$$

In accordance with Eq.(3), applying the inverse Fourier transform in Eq.(32) yields

$$f_1(u) = \mathcal{L}^{-1}\left\{F_1(s)\right\} = -\delta(u) , \quad u\to\infty , \qquad (33)$$

where $\delta(u)$ denotes the Dirac delta function. Finally, applying Eq.(2) in Eq.(33) yields

$$f(u) = 1 + \int_0^t f_1(\xi)\, d\xi = 1 - h(u) = 0 , \quad u\to\infty , \qquad (34)$$

where $h(u)$ is the Heaviside unit-step function. Thus it can be concluded that the function of the initial charge distribution satisfies the features given by Eqs (7), (8) and (9). Its behaviour is not affected by parameters λ_{di}, $i = 1,2,...$ although their values in the general case are not zero.

6. Conclusion

The behaviour of the channel discharge function introduced by the GTCS return stroke model in the complex and time domain are analyzed. Taking into account the constrictions of the GTCS model, the inequalities which have to be satisfied between the parameters of the channel-base current and the initial channel charge distribution function are derived. The approximative expressions of the channel discharge function in the general case of the GTCS during risetime are also derived. They can be used for the calculation of the lightning current along the channel in a few microseconds of the discharge, the period of time during the magnitude of the LEMP as well as the magnitude of its first derivative is generated. The special cases of the GTCS model (the Bruce-Golde, the Traveling Current Source, the Diendorfer-Uman, the Thottappillil-Uman and the modified Diendorfer-Uman model) are also considered regarding the calculation of the channel discharge function and the lightning current along the channel. Although the current reflection from the bottom of the channel can be taken into account we have neglected it due to the simplicity of the mathematical derivations. The results obtained in the paper will provide an easy calculation of the channel discharge function based on the LEMP and the channel-base current measurements. Therefore they will enable better examination of the dynamics of the internal gaseous-physical processes in the lightning channel during the return stroke.

7. Appendices: The analytical form of different current distributions along the channel using the GTCS

7.1 The TCS model

Substituting $\lambda_{di} = 0$, $i = 1,2,...,$ $n = m$, $\lambda_1 = \tau_1 v^*$, $\lambda_2 = \tau_2 v^*$ into Eqs (10) and (11) from Eq.(12) one obtains

$$Q_{01}^{/} = I_0/(\eta\, v^*) . \qquad (35)$$

From Eqs (13) and (14) we get

$$F_1(s) = -[I_0/(Q_{01}^{/}\eta)]\,[L(s)/G(s/v^*)] = -1 . \qquad (36)$$

where $G(s/v^*) = v^* L(s)$. Applying Eqs (3) and (2) we arrive at

$$f_1(u) = \mathscr{F}^{-1}\left\{F_1(s)\right\} = -\delta(u) , \qquad (37)$$

$$f(u) = 1 - h(u) , \quad u = t - z/v . \qquad (38)$$

In further derivations we use the sampling feature of the Dirac function

$$\int_{-\infty}^{+\infty} \delta(t_0 - \xi)\, f(\xi)\, d\xi = f(t_0) . \qquad (39)$$

Using Eqs (5) and (11) the current along the channel will be

$$\begin{aligned} i(z,t) &= \int_z^{h_{az}(z,t)} q_0^{/}(\xi)\, (\partial/\partial t) f(t - \xi/v^* + z/c)\, d\xi \\ &= (I_0/\eta) \int_{z/v^*}^{t+z/c} g(\xi^{/})\ \delta(t - \xi^{/} + z/c)\ d\xi^{/} \\ g(\xi^{/}) &= [\xi^{/\,n}/(\tau_1^n + \xi^{/\,n})] \exp(-\xi^{/}/\tau_2) , \ \xi^{/} = \xi/v^* . \end{aligned} \qquad (40)$$

Using Eq (39) it follows

$$i(z,t) = (\mathrm{I}_0/\eta)\ g(t+z/v^*)$$
$$= \frac{\mathrm{I}_0}{\eta}\ \frac{(t+z/c)^n}{\tau_1^n + (t+z/c)^n}\exp\left(-\frac{t+z/c}{\tau_2}\right) = i_0(t+z/c) \quad , \tag{41}$$

where i_0 is the channel-base current given by Eq.(10). This expression represents the lightning current along the channel according to the TCS model [2]. At the altitude of the return stroke wavefront ($t = z/v$) the current and its derivative have the values which are not equal to zero

$$i(z,t{=}z/v) \neq 0 \ , \quad di(z,t)/dt|_{\,t=z/v} \neq 0 \ . \tag{42}$$

Therefore these discontinuities cause similar discontinuities in the calculated LEMP as well as in field derivative (in the radiation or far field component) [see ref.6]

The BG model will not be separately derived because it can be treated as the special case of the TCS model if $c \to \infty$. Hence from Eq.(41) it follows $i(z,t) = i_0(t)$.

7.2 The DU model

Substituting $\lambda_{d1} = v^*\tau_d, \lambda_{di} = 0,\ \mathrm{i} = 2,3,...$ (there are two constants in the DU model, the breakdown τ_{dbd} and the corona τ_{dco} time discharge constant, [3]; in further derivations we shall denote any of these constants as τ_d) $n = m$, $\lambda_1 = \tau_1 v^*$, $\lambda_2 = \tau_2 v^*$ into Eqs (10) and (11) from Eq.(12) one obtains the equal result as in the case of the TCS model

$$Q_{01}^{\prime} = \mathrm{I}_0/(\eta\, v^*) \ . \tag{43}$$

From Eqs (13) and (14) we get

$$\mathrm{F}_1(s) = -1/(1+\tau_d s) \ . \tag{44}$$

Using a similar procedure as in the previous case we obtain

$$f_1(u) = \mathscr{F}^{-1}\left\{\mathrm{F}_1(s)\right\} = -(1/\tau_d)\exp(-u/\tau_d) \ , \tag{45}$$

The current along the channel will be

$$i(z,t) = \int_z^{h_{az}(z,t)} q_0^{\prime}(\xi)\,\frac{\partial}{\partial t} f(t-\xi/v^* + z/c)\, d\xi$$
$$= \frac{\mathrm{I}_0}{\eta\,\tau_\mathrm{d}}\int_{z/v^*}^{t+z/c}\left[g(\xi^{\prime}) + \tau_d\frac{dg(\xi^{\prime})}{d\xi^{\prime}}\right]\exp\left[-\frac{t-\xi^{\prime}+z/c}{\tau_d}\right]d\xi^{\prime} \tag{46}$$
$$g(\xi^{\prime}) = [\xi^{\prime\,n}/(\tau_1^n + \xi^{\prime\,n})]\exp(-\xi^{\prime}/\tau_2) \ , \ \xi^{\prime} = \xi/v^* \ .$$

By partial integration of the second term in Eq.(46) we have

$$i(z,t) = (\mathrm{I}_0/\eta)\ g(\xi^{\prime})\ \exp(t-\xi^{\prime}+z/c)/\tau_d\ \Big|_{z/v^*}^{t+z/c}$$
$$= \frac{\mathrm{I}_0}{\eta}\left\{\frac{(t+z/c)^n}{\tau_1^n+(t+z/c)^n}\exp[-(t+z/c)/\tau_2] - \right.$$
$$\left. - \frac{(z/v^*)^n}{\tau_1^n + (z/v^*)^n}\exp[-(z/v^*)/\tau_2]\ \exp[-(t-z/v)/\tau_d]\right\} \tag{47}$$
$$= i_0(t+z/c) - i_0(z/v^*)\ \exp[-(t-z/v)/\tau_d] \ .$$

Eq.(47) provides the expression of the lightning current along the channel according to the DU model [3]. At the altitude of the return stroke wavefront the current has no discontinuities but there is a discontinuity of the current derivative

$$i(z,t{=}z/v) = 0 \ , \quad di(z,t)/dt|_{\,t=z/v} \neq 0 \ . \tag{48}$$

7.3 The TU model

In the TU model [4] there is only one height-dependent time discharge constant $\tau_d(z)$. Its value can be calculated from the simultaneous measurement of the channel-base current and the radiated LEMP. The procedure for finding the channel discharge function is similar as in the case of the DU model. Thus one should accept the following values of the parameters in Eqs (10) and (11): $n = m$, $\lambda_1 = \tau_1 v^*, \lambda_2 = \tau_2 v^*$ and $\lambda_{d1}(z) = v^*\tau_d(z)$, $\lambda_{di} = 0$, $\mathrm{i} = 2,3,...$ From Eq.(12) we get the same result given by Eq.(43). From Eqs (13) and (14) we have

$$\mathrm{F}_1(s) = -1/[1+\tau_d(z)\,s] \ . \tag{49}$$

Using a similar procedure as in the cases of the TCS and DU models we obtain

$$f_1(z,u) = \mathscr{F}^{-1}\left\{\mathrm{F}_1(s)\right\} = -\tau_d(z)^{-1}\exp[-u/\tau_d(z)] \ , \tag{50}$$

In this case it is not possible to give the analytical form of the lightning current along the channel as it is done in the cases of the TCS and the DU model. This can be easily understood because the shape of the function $\tau_d(z)$ is not given beforehand. Hence the expression of the current along the channel will be

$$i(z,t) = \int_z^{h_{az}(z,t)} q_0^{\prime}(\xi)\,\frac{\partial}{\partial t} f(z, t-\xi/v^* + z/c)\, d\xi$$
$$= \frac{\mathrm{I}_0}{\eta}\int_{z/v^*}^{t+z/c}\left[\frac{g(\xi^{\prime})}{\tau_d(v^*\xi^{\prime})} + \frac{dg(\xi^{\prime})}{d\xi^{\prime}}\right] \times$$
$$\times\ \exp\left(-\frac{t-\xi^{\prime}+z/c}{\tau_d(v^*\xi^{\prime})}\right)d\xi^{\prime} \ , \tag{51}$$
$$g(\xi^{\prime}) = \frac{\xi^{\prime\,n}}{\tau_1^n + \xi^{\prime\,n}}\exp(-\xi^{\prime}/\tau_2) \quad , \quad \xi^{\prime} = \xi/v^* \quad .$$

In order to calculate the current along the channel, Thottappillil and Uman [4] divided the activated length of the channel in N sections, assuming the constant value for τ_d in each of them. The final shape of the function $\tau_d(z)$ was result of the best matching of the calculated with the measured LEMP. It means that the integral given by Eq.(51) can be approximately solved in the same manner, by dividing it into N addends. This gives the idea for further calculations using the GTCS if the current integral, Eq.(5) could not be analytically solved.

8. References

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